Approximation Algorithms for the Unsplittable Capacitated Facility Location Problem

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July 5, 2012
Problem Statement

Unsplittable Capacitated Facility Location (UCFL) Problem

- **Input:** $F =$ set of facilities and $C =$ set of clients, a metric cost function $c$ between $F$ and $C$, demand of client $j = d_j$, opening cost of facility $i = f_i$.
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- **Extra Input:** capacity of facility $i = u_i$
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- **Extra Input:** capacity of facility $i = u_i$
- **Constraints:** unsplittable demand, do not violate capacities.
An Example of UCFL

All the other cost values are equal to the shortest path value in the above graph, e.g., \( c_{31} = 4 \).
An Example of UCFL

Solution 1: Open the second and third facilities. Service cost is 18, facility cost is 3 and total cost is 21.
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**Solution 1:** Open the second and third facilities. Service cost is 18, facility cost is 3 and total cost is 21.

**Solution 2:** Open the first and fourth facilities. Service cost is 16, facility cost is 11 and total cost is 27.
Motivations

Original Motivation

Location Problems in the operation research
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Location Problems in the operation research

New motivation

Contents Distribution Networks (CDNs):

- Alzoubi et al. (WWW ’08): A load-aware IP Anycast CDN architecture
- The assignment of downloadable objects, such as media files, to some servers
Solving the UCFL problem without violation of capacities is $NP$-hard.
Solving the UCFL problem without violation of capacities is \( NP \)-hard.

\((\alpha, \beta)\)-approximation algorithm for the UCFL problem: cost within factor \( \alpha \) of the optimum, violates the capacity constraints within factor \( \beta \).
Related Works to Variations of UCFL

- Uncapacitated Facility Location Problem
  - current best approximation ratio = 1.488 (Li, ICALP’11)
  - current best hardness ratio = 1.463 (Guha-Khuller, SODA’98 + Sviridenko’s observation)
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- **Splittable Capacitated Facility Location Problem**
  - current best approximation ratio = 5.83 (or 5?) in the non-uniform case (Zhang-Chen-Ye, Mathematics of OR’05) and 3 in the uniform case (Aggarwal et al., IPCO’10)
  - current best hardness ratio = 1.463
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- $(11, 2)$ for non-uniform case and $(5, 2)$ for uniform case
**UCFL Previous Results**

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**Algorithmic Results:**
The first approximation algorithm: \((9, 4)\)-approximation for the uniform case (Shmoys-Tardos-Aardal, STOC’97.)

Current best approximation algorithms:
- \((11, 2)\) for non-uniform case and \((5, 2)\) for uniform case
- uniform case: \((O(\log n), 1 + \epsilon)\) for any \(\epsilon > 0\) in polynomial time (Bateni-Hajiaghayi, SODA’09.)
- non-uniform case: \((O(\log n), 1 + \epsilon)\) for any \(\epsilon > 0\) in quasi-polynomial time (Bateni-Hajiaghayi, SODA’09.)
Recall: The best possible is $(O(1), 1 + \epsilon)$-approximation unless $P = NP$. 
New Results

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We only consider the **uniform** case.

All capacities are uniform \(\rightarrow\) we can assume that \(u = 1\) and \(d_j \leq 1\) for all \(j \in C\).
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**Definition**

An \(\epsilon\)-restricted UCFL, denoted by \(\text{RUCFL}(\epsilon)\), instance is an instance of the UCFL in which \(\epsilon < d_j \leq 1\) for all \(j \in C\).
New results, Cont’d

**Theorem**

*(Weaker Version)* If $A$ is an $(\alpha, \beta)$-approximation algorithm for the $RUCFL(\epsilon)$ then there is an algorithm $A_C$ for UCFL with factor

$$(10\alpha + 11, \max\{\beta, 1 + \epsilon\}).$$
New results, Cont’d

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$$(10\alpha + 11, \max\{\beta, 1 + \epsilon\}).$$

Corollary

For any constant $\epsilon > 0$, an $(O(1), 1 + \epsilon)$-approximation algorithm for the $RUCFL(\epsilon)$ yields an $(O(1), 1 + \epsilon)$-approximation for the $UCFL$. 
New Results, Cont’d

Theorem

There is a polynomial time $(10.173, 3/2)$-approximation algorithm for the UCFLP.

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Theorem

There exists a \((1 + \epsilon, 1 + \epsilon)\)-approximation algorithm for the Euclidean UCFL in \(\mathbb{R}^2\) with running time in quasi-polynomial for any constant \(\epsilon > 0\).
Some More Definitions

- **Large** clients = clients with demand more than $\epsilon$,
  $L = \{j \in C : d_j > \epsilon\}$. 
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- **Small** clients = clients with demand at most $\epsilon$, $S = C \setminus L$. 

OPT = optimum value
Some More Definitions

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- **Small** clients = clients with demand at most $\epsilon$, $S = C \setminus L$.
- $\phi_1 : C_1 \rightarrow F_1$ and $\phi_2 : C_2 \rightarrow F_2$ are consistent if $\phi_1(j) = \phi_2(j)$ for all $j \in C_1 \cap C_2$. 

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Proof of Reduction to RUCFL
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   1. Run \( A \) to assign large clients.
   2. For opened facilities, set \( f_i = 0 \) and set \( u'_i \) to unused capacity of facility \( i \).
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1. Assign small clients \textit{fractionally} by an approximation algorithm for the splittable CFLP.

2. Assign small clients \textit{integraally}: round the splittable assignment by Shmoys-Tardos algorithm for the Generalized Assignment Problem.
Basic idea: Ignoring small clients in step 1 is not a big mistake!
Proof of Reduction to RUCFL, Cont’d

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**Lemma**

(Weaker Version) There exist a fractional assignment of small clients with service cost at most $(\alpha + 1)OPT$ and facility cost at most $OPT$.

Splitable CFLP algorithm $\rightarrow$ finds a fractional assignment having cost within constant factor of this fractional assignment.
Proof of Reduction to RUCFL, Cont’d

$s_i = \text{total demand of small clients assigned to } i\text{th facility}$

- $s_1 = 9$
- $s_2 = 5$
- $s_3 = 3$
- $s_4 = 2$

**General Idea:** Change an optimal solution to a solution consistent with our assignment.
Proof of Reduction to RUCFL, Cont’d

- General Idea: Change an optimal solution to a solution consistent with our assignment.
- Switch the assignment of large clients one by one.
- service cost \(\leq\) service cost of small clients in optimum plus service cost of large clients in optimum (OPT) plus service cost of large clients \(\alpha\text{OPT}\).
Proof of Reduction to RUCFL, Cont’d

\[ d_2 = 3 \quad d_4 = 8 \]

\[ s_1 = 4 \quad s_2 = 10 - 3 = 7 \quad s_3 = 3 + 3 = 6 \quad s_4 = 2 \]

\( F \)

\( C \)

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Proof of Reduction to RUCFL, Cont’d

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- **General Idea:** Change an optimal solution to a solution consistent with our assignment.
- **Switch** the assignment of large clients one by one. **Order?**
- **service cost \( \leq \)** service cost of small clients in optimum plus service cost of large clients in optimum \((OPT)\) plus service cost of large clients \(\alpha OPT\).
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Proof of Reduction to RUCFL, Cont’d

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Switch the assignment of large clients one by one. Order?

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Do all switches simultaneously.
We showed there is a fractional assignment of small clients with low cost.

We found one with a low cost by an approximation algorithm. Now?
Proof of Reduction to RUCFL, Cont’d

- We showed there is a fractional assignment of small clients with low cost.
- We found one with a low cost by an approximation algorithm. Now?
- Using rounding for Generalized Assignment problem:
  - Connection cost remains the same.
  - It violates the capacities at most to the extent of the largest demand.
  - The largest demand is at most $\epsilon \rightarrow$ violation is within factor $1 + \epsilon$. 
RUCFL(\(\frac{1}{2}\))

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proof

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- The algorithm is a min-cost maximum matching algorithm.
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Corollary

There is a (10.173, 3/2)-approximation algorithm for the UCFL problem.
To solve the UCFL problem, we transformed the problem to a simpler version.
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We solved the simpler version for $\epsilon = 1/2$ and $\epsilon = 1/3$ to obtain factor $(10.173, 3/2)$ and $(30.432, 4/3)$ approximation algorithms.
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We solved the simpler version for $\epsilon = \frac{1}{2}$ and $\epsilon = \frac{1}{3}$ to obtain factor $(10.173, \frac{3}{2})$ and $(30.432, \frac{4}{3})$ approximation algorithms.

Open question? Finding a $(O(1), 1 + \epsilon)$-approximation algorithm for the UCFL problem.
Thanks for your attention!
Questions?