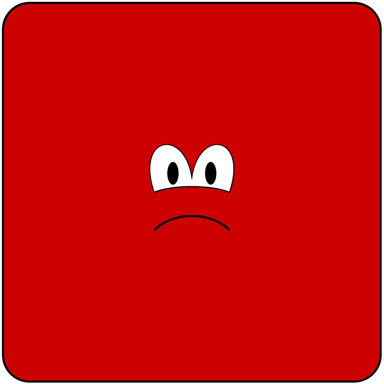
Kernel lower bounds using co-nondeterminism: Finding induced hereditary subgraphs

Stefan Kratsch, Marcin Pilipczuk, Ashutosh Rai, Venkatesh Raman

SWAT 2012, Helsinki

Kernelization



instance of NP-hard problem

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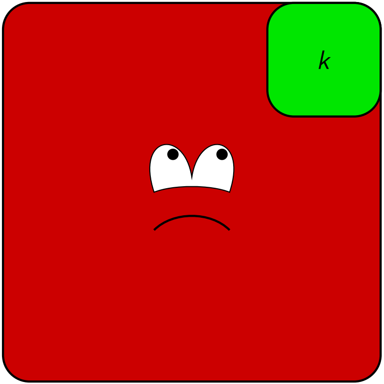

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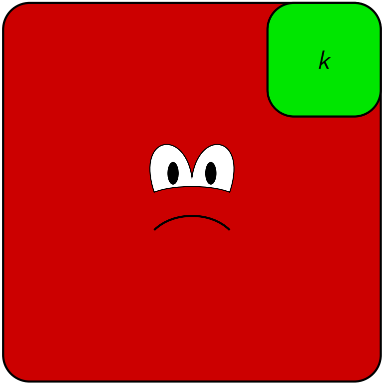

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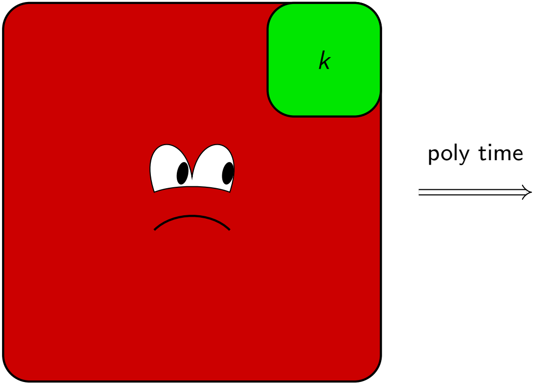

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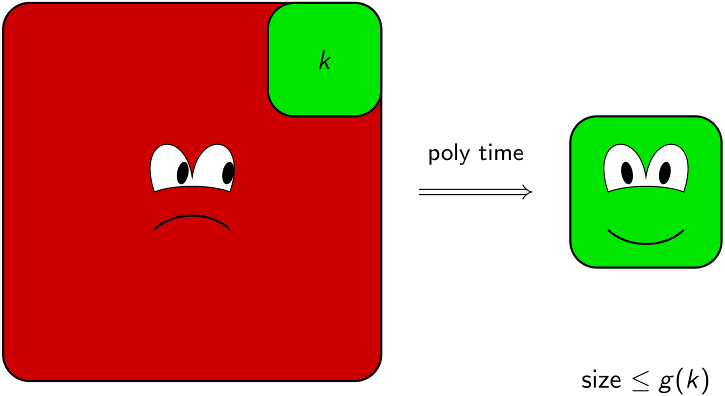

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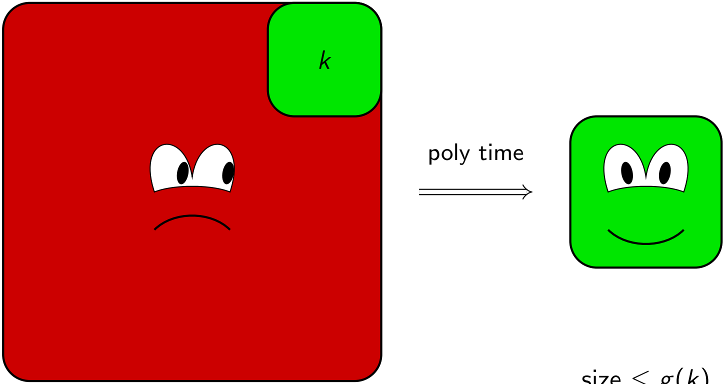

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size $\leq g(k)$
small: $g(k)$ polynomial

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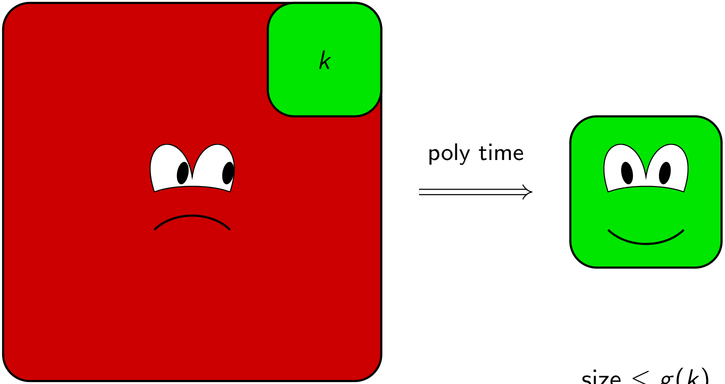

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sometimes impossible

Kernelization lower bounds
[Bodlaender, Downey, Fellows, Hermelin, ICALP'08]

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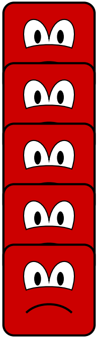

t instances
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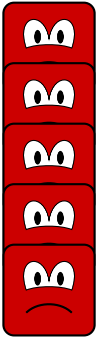

poly time

$\Longrightarrow$

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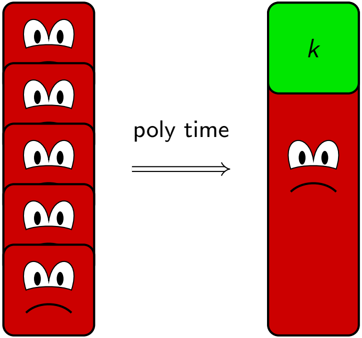

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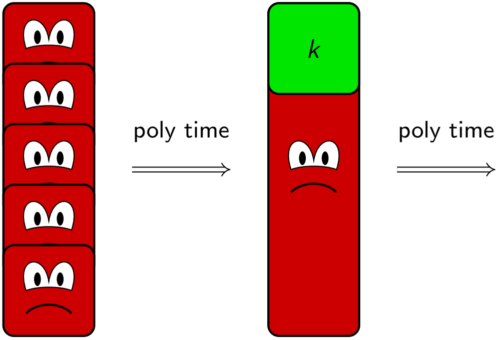

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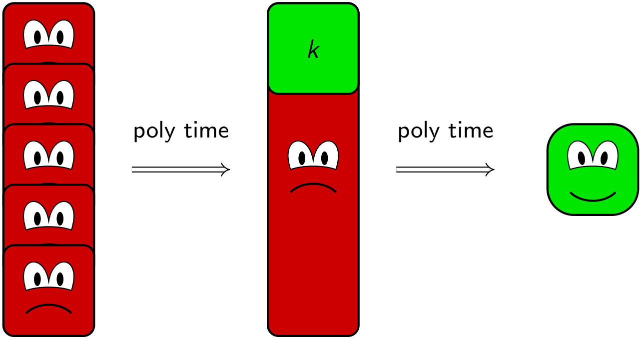

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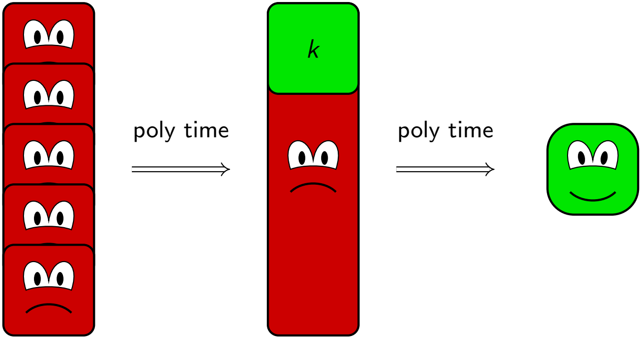

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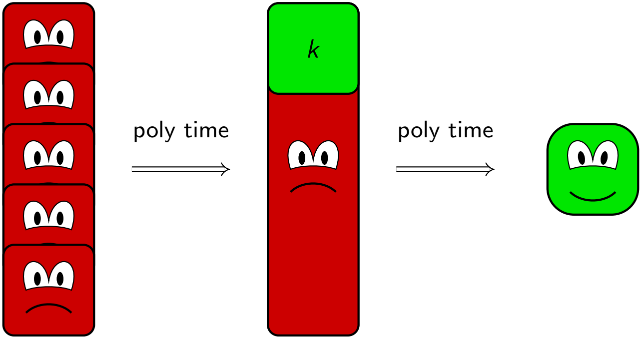

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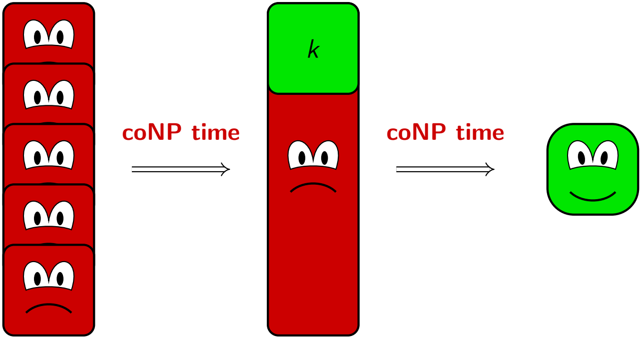

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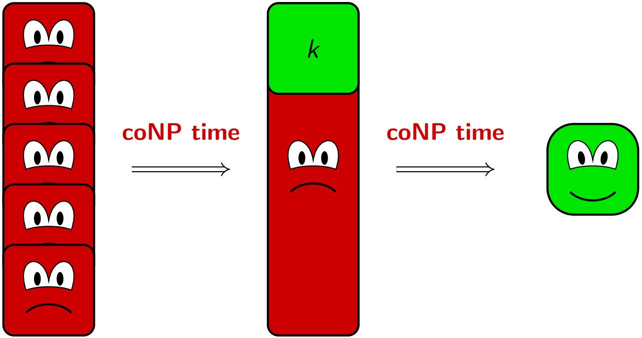

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under "coNP reductions"

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  - That is, if at least one $x_i \in L$, then **all** outputs $\in Q$.
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- Then, a (co-nondeterministic) polynomial kernelization of Q implies $\mathrm{NP} \subseteq \mathrm{coNP}/\mathrm{poly}$.

Case study: Π-INDUCED SUBGRAPH

- Co-nondeterminism first used by Kratsch (SODA'12) for RAMSEY.

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  - Note: we need Π to be poly-recognizable.

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co-nondeterministic compositions

Assume Π is **closed under embedding**.

Π-Induced Subgraph
co-nondeterministic compositions

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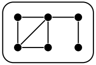
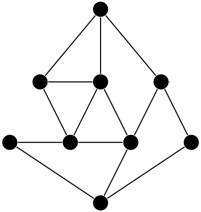
H





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Π-Induced Subgraph
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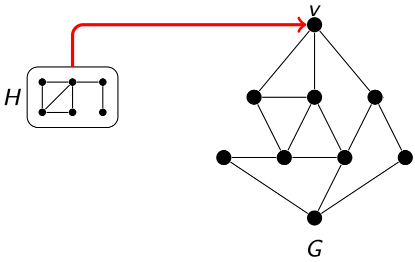

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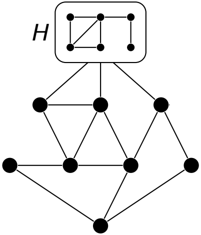

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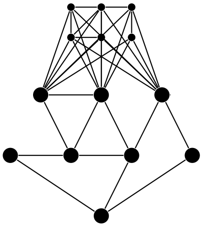

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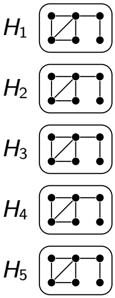

t instances of
Improvement-Π-IS
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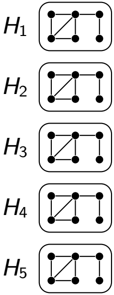

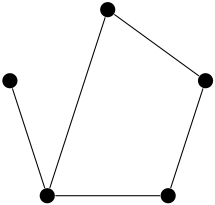

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host graph with t vertices

Π-INDUCED SUBGRAPH
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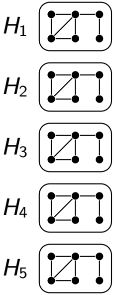

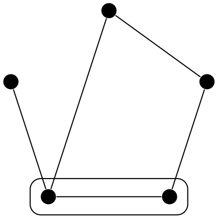

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each vertex $\in$ ℓ-vertex Π-ind. subgr.

Π-Induced Subgraph

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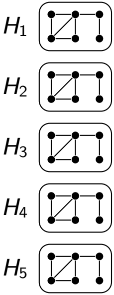

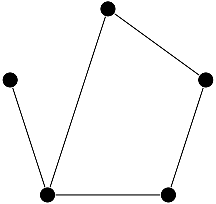

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Π-Induced Subgraph

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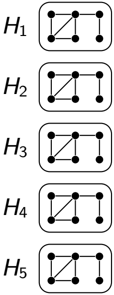

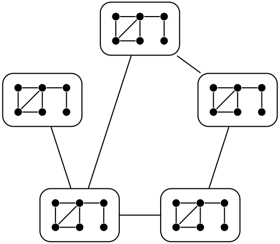

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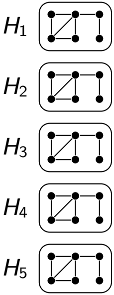

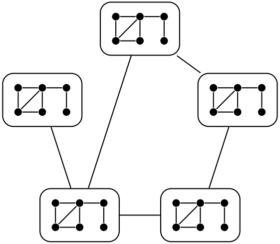

$k' = \ell(k-1)+1$

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- As in [Kratsch, SODA 2012]: use co-nondeterminism!

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- Π has the Erdős-Hajnal property $\Rightarrow$ good host graph exists and we can find it in coNP-time.

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Note: excluding a biclique implies the Erdős-Hajnal property.

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- Big open problem: prove Erdős-Hajnal conjecture.

Thank you

Questions?

Tikz faces based on a code by Raoul Kessels,
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