Probabilistic Analysis of Christofides' Algorithm

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Stochastic Euclidean TSP

Problem

Given n points $a_1, \ldots a_n$ from $[0,1]^d$, compute the shortest travelling salesman's tour $T(a_1, \ldots, a_n)$.

Stochastic Euclidean TSP

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Given n points $a_1, \ldots a_n$ from $[0,1]^d$, compute the shortest travelling salesman's tour $T(a_1, \ldots, a_n)$.

- NP hard to compute exactly.
- PTAS algorithms are known. [Arora '96, Mitchell '99]

Probabilistic Analysis of Stochastic ETSP

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Given $X_1, \ldots, X_n$ uniform, i.i.d points from $[0,1]^d$, provide a.s. theory for $T(X_1, \ldots, X_n)$.

Probabilistic Analysis of Stochastic ETSP

Problem

Given $X_1, \ldots, X_n$ uniform, i.i.d points from $[0,1]^d$, provide a.s. theory for $T(X_1, \ldots, X_n)$.

Theorem (Beardwood-Halton-Hammersly '59)

There exists a positive constant $\alpha(d)$ such that,

$$\lim_{n\to\infty} \frac{T(X_1, \ldots, X_n)}{n^{(d-1)/d}} = \alpha(d) \text{ with probability one.}$$

Probabilistic Analysis of Stochastic ETSP

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Question[Frieze-Yukich 2000] Develop *a.s* theory for the Christofides' algorithm.

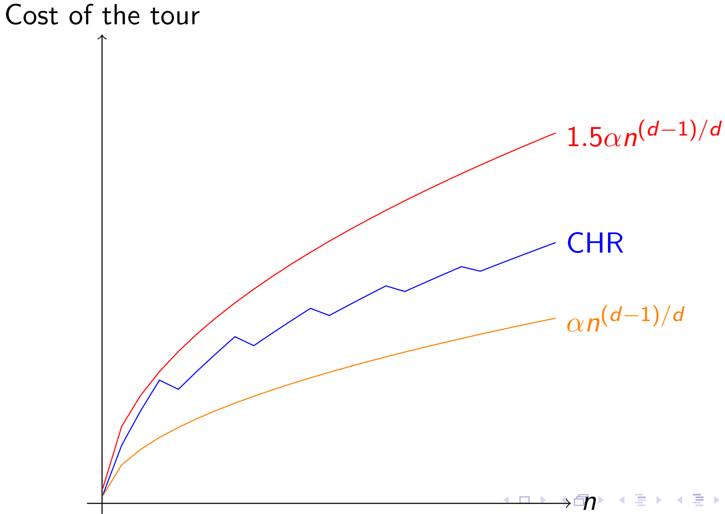
Christofides' algorithm for Stochastic ETSP

- Compute a minimum spanning tree τ of the given set of points $a_1 \ldots, a_n \in [0,1]^d$.
- Let M be minimum matching of the odd-degree vertices in τ and $G = \tau \cup M$.
- Output the tour obtained by short-cutting the Eulerian graph. G.

Christofides' algorithm for Stochastic ETSP

- Christofides' algorithm has a worst-case approximation ratio of 1.5.
- The ratio is tight for Euclidean Metric.
- Experiments suggest better performance in practice.

Probabilistic Analysis?



Christofides' functional

Definition

For $F \subset [0,1]^d$ with $|F| = n$,

$$\mathrm{CHR}(F) \triangleq \mathrm{MST}(F) + \text{ODD-MATCHING}(F).$$

Main Theorem

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There exists a positive constant $\beta(d)$ such that,

$$\lim_{n\to\infty} \frac{\mathbb{E}[\mathrm{CHR}(X_1,\ldots,X_n)]}{n^{(d-1)/d}} = \beta(d)$$

where $X_1,\ldots,X_n$ are independent unform distributions from $[0,1]^d$.

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where $X_1,\ldots,X_n$ *are independent unform distributions from* $[0,1]^d$*.*

Corollary

There is positive constant $\beta(d)$ *such that,*

$$\lim_{n\to\infty} \frac{[\mathrm{CHR}(X_1,\ldots,X_n)]}{n^{(d-1)/d}} = \beta(d) \textit{ with probability one.}$$

Geometric Subadditivity

Definition (Geometric Subbadditivity)

Let $Q_1, \ldots, Q_{m^d}$ be a partition of $[0,1]^d$ into equi-sized sub-cubes of side m^{-1}. A functional f is geometric subbadditive if for all $F \subset [0,1]^d$ and $m > 0$,

$$f(F, [0,1]^d) \leq \sum_{i=1}^{m^d} f(F \cap Q_i, Q_i) + Cm^{d-1}$$

where C is a constant depending on d.

Geometric Subadditivity

- The functionals corresponding to Euclidean TSP, Euclidean MST and Euclidean minimum matching are geomtric subadditive.

Limit theorems for Subadditive functionals

Theorem (Steele '81)

If f is a monotone and subadditive Euclidean functional over $[0,1]^d$, then there is a constant $\alpha_f(d)$ such that,

$$\lim_{n\to\infty} \frac{f(X_1,\ldots,X_n)}{n^{(d-1)/d}} = \alpha_f(d) \text{ with probability one}$$

where $X_1,\ldots,X_n$ are independent uniform distributions over $[0,1]^d$.

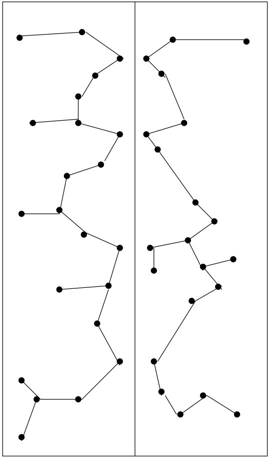
Limit theorems for Subadditive functionals

- CHR is not monotone.

Limit theorems for Subadditive functionals

- CHR is not monotone.
- Assumption of montonicity can be removed from Steele's theorem. [Yukich '96]

Is CHR subadditive?



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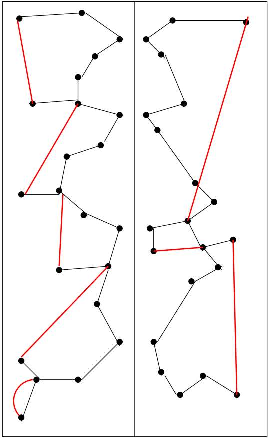

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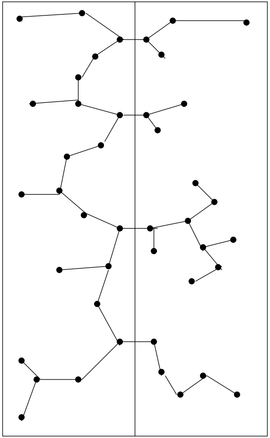

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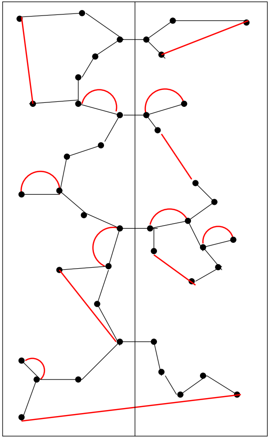

Weak Subadditivity

Definition (Weak Subbadditivity)

Let $Q_1, \ldots, Q_{m^d}$ be a partition of $[0,1]^d$ into equi-sized sub-cubes of side m^{-1}. A functional f is weakly subbadditive if for all $F \subset [0,1]^d$ and $m > 0$,

$$f(F, [0,1]^d) \leq \sum_{i=1}^{m^d} f(F \cap Q_i, Q_i) + Cm^{d-1} + o(n^{(d-1)/d})$$

where C is a constant depending on d.

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where C is a constant depending on d.

- Steele's theorem can be extended to functions that are weakly-subadditive. [Golin '96, Baltz et. al '05]

CHR is weakly subadditive

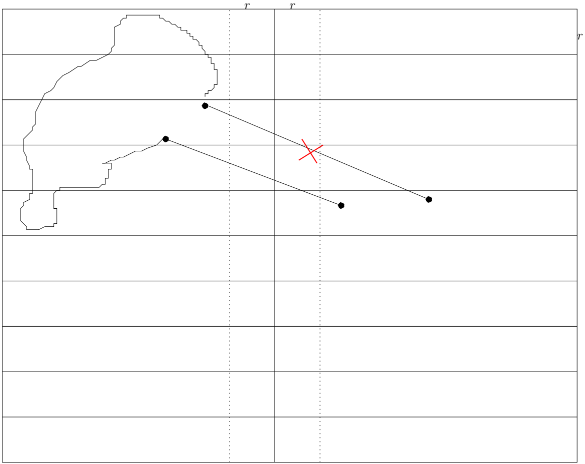
Lemma

CHR *is weakly subaddtitive for* $m < n^{1/(2d)}$

Proof Sketch

- The total cost of MST edges that cross the boundary of sub-cubes $Q_1, \ldots, Q_{m^d}$ is small.

Proof Sketch



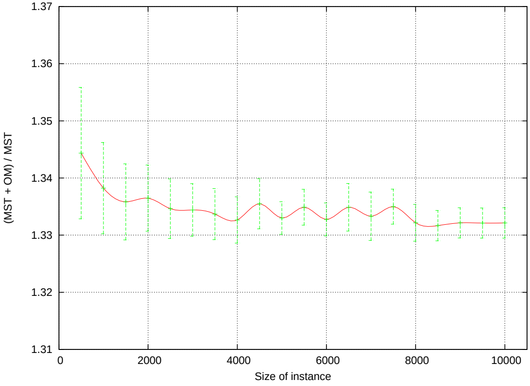
Proof Sketch

- The total cost of MST edges that cross the boundary of sub-cubes $Q_1, \ldots, Q_{m^d}$ is small.
- The cost of matching edges induced by the new boundary edges can be bounded by that of the boundary edges plus cost of matching in $[0, 1]^{d-1}$

Extensions

- Can be extended to the case of non-uniform distributions using the boundary process approach introduced by Redmond and Yukich.
- Tail bounds can be obtained using Rhee's isoperimetric inequality.[Rhee]

Experimental evaluation of β



Open Questions

- Obtain an estimate for the constant $\beta(d)$.
- Estimate the cost gains made my shortcutting.
- Extend the analysis to the case of non-identical distributions.

THANK YOU